SURFACES WITH PARALLEL MEAN CURVATURE IN $\mathbb{S}^3 \times \mathbb{R}$ AND $\mathbb{H}^3 \times \mathbb{R}$

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Abstract. We prove a Simons type equation for non-minimal surfaces with parallel mean curvature vector (pmc surfaces) in $M^3(c) \times \mathbb{R}$, where $M^3(c)$ is a 3-dimensional space form. Then, we use this equation in order to characterize certain complete non-minimal pmc surfaces.

1. Introduction

In 1968, J. Simons discovered a fundamental formula for the Laplacian of the second fundamental form of a minimal submanifold in a Riemannian manifold. He then used it to characterize certain minimal submanifolds of a sphere and Euclidean space (see [16]). One year later, K. Nomizu and B. Smyth generalized Simons' equation for hypersurfaces of constant mean curvature (cmc hypersurfaces) in a space form (see [14]). This was extended, in B. Smyth's work [17], to the more general case of a submanifold with parallel mean curvature vector (pmc submanifold) in a space form. Over the years such equations, called Simons type equations, turned out to be very useful, a great number of authors using them in the study of cmc and pmc submanifolds (see, for example, [3, 9, 15]).

Nowadays, the study of cmc surfaces in the Euclidean space and, more generally, in space forms is a classical subject in the field of Differential Geometry, well known papers like [12] by H. Hopf and [8] by S.-S. Chern being representative examples for the literature on this topic. When the codimension is greater than 1, a natural generalization of cmc surfaces are pmc surfaces, which have been intensively studied in the last four decades, among the first papers devoted to this subject being [10] by D. Ferus, [7] by B.-Y. Chen and G. D. Ludden, [11] by D. A. Hoffman and [18] by S.-T. Yau. All results in these papers were obtained in the case when the ambient space is a space form.

The next natural step was taken by U. Abresch and H. Rosenberg, who studied in [1, 2] cmc surfaces and obtained Hopf type results in product spaces $M^2(c) \times \mathbb{R}$, where $M^2(c)$ is a complete simply-connected surface with constant curvature c, as well as the homogeneous 3-manifolds $Nil(3)$, $\widetilde{PSL}(2, \mathbb{R})$ and Berger spheres. Some of their results in [1] were extended to pmc surfaces in product spaces of $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is an n-dimensional space form, in [4, 5] by H. Alencar, M. do Carmo and R. Tribuzy.

In a recent paper M. Batista derived a Simons type equation, involving the traceless part of the second fundamental form of a cmc surface in $M^2(c) \times \mathbb{R}$ (see [6]), and found several applications.

2000 *Mathematics Subject Classification.* 53A10, 53C42.

Key words and phrases. surfaces with parallel mean curvature vector, Simons type equation.

The first author was supported by a Post-Doctoral Fellowship "Pós-Doutorado Sênior (PDS)" offered by FAPERJ, Brazil.

In this paper we compute the Laplacian of the squared norm of the traceless part ϕ of the second fundamental form σ of a pmc surface in a product space $M^3(c)\times\mathbb{R}$ and, using this Simons type formula, we characterize some of these surfaces. The main results of our paper are the following.

Theorem 5.2. *Let Σ^2 be an immersed pmc 2-sphere in $M^n(c)\times\mathbb{R}$, such that*

(1) $|T|^2=0$ *or* $|T|^2\geq\frac{2}{3}$ *and* $|\sigma|^2\leq c(2-3|T|^2)$, *if* $c<0$;
(2) $|T|^2\leq\frac{2}{3}$ *and* $|\sigma|^2\leq c(2-3|T|^2)$, *if* $c>0$,

where T is the tangent part of the unit vector ξ tangent to $\mathbb{R}$. Then, Σ^2 is either a minimal surface in a totally umbilical hypersurface of $M^n(c)$ or a standard sphere in $M^3(c)$.

Theorem 5.3. *Let Σ^2 be an immersed complete non-minimal pmc surface in $\bar{M}=M^3(c)\times\mathbb{R}$, with $c>0$ and mean curvature vector H. Assume*

i) $|\phi|^2\leq 2|H|^2+2c-\frac{5c}{2}|T|^2$, *and*
ii) a) $|T|=0$, *or*
  b) $|T|^2>\frac{2}{3}$ *and* $|H|^2\geq\frac{c|T|^2(1-|T|^2)}{3|T|^2-2}$.

Then either

(1) $|\phi|^2=0$ *and Σ^2 is a round sphere in $M^3(c)$, or*
(2) $|\phi|^2=2|H|^2+2c$ *and Σ^2 is a torus* $\mathbb{S}^1(r)\times\mathbb{S}^1(\sqrt{\frac{1}{c}-r^2})$, $r^2\neq\frac{1}{2c}$, *in* $M^3(c)$.

2. Preliminaries

Let $M^n(c)$ be a simply-connected n-dimensional manifold, with constant sectional curvature c, and consider the product manifold $\bar{M}=M^n(c)\times\mathbb{R}$. The expression of the curvature tensor $\bar{R}$ of such a manifold can be obtained from

$$\langle\bar{R}(X,Y)Z,W\rangle=c\{\langle d\pi Y,d\pi Z\rangle\langle d\pi X,d\pi W\rangle-\langle d\pi X,d\pi Z\rangle\langle d\pi Y,d\pi W\rangle\},$$

where $\pi:\bar{M}=M^n(c)\times\mathbb{R}\to M^n(c)$ is the projection map. After a straightforward computation we get

$$\begin{aligned}\bar{R}(X,Y)Z=\ & c\{\langle Y,Z\rangle X-\langle X,Z\rangle Y-\langle Y,\xi\rangle\langle Z,\xi\rangle X+\langle X,\xi\rangle\langle Z,\xi\rangle Y\\ &+\langle X,Z\rangle\langle Y,\xi\rangle\xi-\langle Y,Z\rangle\langle X,\xi\rangle\xi\},\end{aligned}\tag{2.1}$$

where ξ is the unit vector tangent to $\mathbb{R}$.

Now, let Σ^2 be an immersed surface in $\bar{M}$, and denote by R its curvature tensor. Then, from the equation of Gauss

$$\begin{aligned}\langle R(X,Y)Z,W\rangle=\ &\langle\bar{R}(X,Y)Z,W\rangle\\ &+\textstyle\sum_{\alpha=3}^{n+1}\{\langle A_\alpha Y,Z\rangle\langle A_\alpha X,W\rangle-\langle A_\alpha X,Z\rangle\langle A_\alpha Y,W\rangle\},\end{aligned}$$

we obtain

$$\begin{aligned}R(X,Y)Z=\ & c\{\langle Y,Z\rangle X-\langle X,Z\rangle Y-\langle Y,T\rangle\langle Z,T\rangle X+\langle X,T\rangle\langle Z,T\rangle Y\\ &+\langle X,Z\rangle\langle Y,T\rangle T-\langle Y,Z\rangle\langle X,T\rangle T\}\\ &+\textstyle\sum_{\alpha=3}^{n+1}\{\langle A_\alpha Y,Z\rangle A_\alpha X-\langle A_\alpha X,Z\rangle A_\alpha Y\},\end{aligned}\tag{2.2}$$

where T is the component of ξ tangent to the surface, A is the shape operator defined by the equation of Weingarten

$$\bar{\nabla}_X V=-A_V X+\nabla^{\perp}_X V,$$

for any vector field X tangent to Σ^2 and any vector field V normal to the surface. Here $\bar\nabla$ is the Levi-Civita connection on $\bar M$ and $\nabla^\perp$ is the connection in the normal bundle, and $A_\alpha = A_{E_\alpha}$, $\{E_\alpha\}_{\alpha=3}^{n+1}$ being a local orthonormal frame field in the normal bundle.

Definition 2.1. If the mean curvature vector H of the surface Σ^2 is parallel in the normal bundle, i.e. $\nabla^\perp H = 0$, then Σ^2 is called a *pmc surface*.

We will end this section by recalling the Omori-Yau Maximum Principle, which shall be used later.

Theorem 2.2 ([19]). *If M is a complete Riemannian manifold with Ricci curvature bounded from below, then for any smooth function $u \in C^2(M)$ with $\sup_M u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset M$ satisfying*

$$\lim_{k\to\infty} u(p_k) = \sup_M u, \quad |\nabla u|(p_k) < \frac{1}{k} \quad \text{and} \quad \Delta u(p_k) < \frac{1}{k}.$$

3. A formula for pmc surfaces in $M^n(c)\times\mathbb{R}$

Let Σ^2 be an immersed surface in $M^n(c)\times\mathbb{R}$, with mean curvature vector H. In this section we shall prove a formula for the Laplacian of the squared norm of A_V, where V is a normal vector to the surface, such that V is parallel in the normal bundle, i.e. $\nabla^\perp V = 0$, and $\operatorname{trace} A_V = \text{constant}$.

Lemma 3.1. *If U and V are normal vectors to the surface and V is parallel in the normal bundle, then $[A_V, A_U] = 0$, i.e. A_V commutes with A_U.*

Proof. The conclusion follows easily, from the Ricci equation,

$$\langle R^\perp(X,Y)V,U\rangle = \langle [A_V,A_U]X,Y\rangle + \langle \bar R(X,Y)V,U\rangle,$$

since $R^\perp(X,Y)V = 0$ and (2.1) implies that $\langle \bar R(X,Y)V,U\rangle = 0$, □

Now, from the Codazzi equation,

$$\begin{aligned}\langle \bar R(X,Y)Z,V\rangle = \; & \langle \nabla^\perp_X\sigma(Y,Z),V\rangle - \langle\sigma(\nabla_XY,Z),V\rangle - \langle\sigma(Y,\nabla_XZ),V\rangle\\ & -\langle \nabla^\perp_Y\sigma(X,Z),V\rangle + \langle\sigma(\nabla_YX,Z),V\rangle + \langle\sigma(X,\nabla_YZ),V\rangle,\end{aligned}$$

where σ is the second fundamental form of Σ^2, we get

$$\begin{aligned}\langle \bar R(X,Y)Z,V\rangle = \; & X(\langle A_VY,Z\rangle) - \langle\sigma(Y,Z),\nabla^\perp_XV\rangle - \langle A_V(\nabla_XY),Z\rangle\\ & -\langle A_VY,\nabla_XZ\rangle - Y(\langle A_VX,Z\rangle) + \langle\sigma(X,Z),\nabla^\perp_YV\rangle\\ & +\langle A_V(\nabla_YX),Z\rangle + \langle A_VX,\nabla_YZ\rangle\\ = \; & \langle(\nabla_XA_V)Y - (\nabla_YA_V)X,Z\rangle,\end{aligned}$$

since $\nabla^\perp V = 0$. Therefore, using (2.1), we obtain

$$(\nabla_XA_V)Y = (\nabla_YA_V)X + c\langle V,N\rangle(\langle Y,T\rangle X - \langle X,T\rangle Y), \tag{3.1}$$

where N is the normal part of ξ.

Next, we have

$$\frac{1}{2}\Delta|A_V|^2 = |\nabla A_V|^2 + \langle\nabla^2A_V,A_V\rangle, \tag{3.2}$$

where we extended the metric $\langle,\rangle$ to the tensor space in the standard way.

In order to calculate the second term in the right hand side of (3.2), we shall use a method introduced in [14].

Let us write

$$C(X,Y)=\nabla_X(\nabla_Y A_V)-\nabla_{\nabla_X Y}A_V,$$

and note that the fact that the torsion of ∇ vanishes, together with the definition of the curvature tensor R on the surface, implies

$$C(X,Y)=C(Y,X)+[R(X,Y),A_V]. \tag{3.3}$$

Next, consider an orthonormal basis $\{e_1,e_2\}$ in $T_p\Sigma^2$, $p\in\Sigma^2$, extend e_1 and e_2 to vector fields E_1 and E_2 in a neighborhood of p such that $\nabla E_i=0$ at p, and let X be a tangent vector field such that $\nabla X=0$. Obviously, we have, at p,

$$(\nabla^2 A_V)X=\sum_{i=1}^{2}\nabla_{E_i}(\nabla_{E_i}A_V)X=\sum_{i=1}^{2}C(E_i,E_i)X.$$

Using equation (3.1), we get, at p,

$$\begin{aligned}
C(E_i,X)E_i &= (\nabla_{E_i}(\nabla_X A_V))E_i-(\nabla_{\nabla_{E_i}X}A_V)E_i\\
&= \nabla_{E_i}((\nabla_X A_V)E_i)-(\nabla_X A_V)(\nabla_{E_i}E_i)\\
&= \nabla_{E_i}((\nabla_{E_i}A_V)X)+c\nabla_{E_i}(\langle V,N\rangle(\langle E_i,T\rangle X-\langle X,T\rangle E_i))
\end{aligned}$$

and then

$$\begin{aligned}
C(E_i,X)E_i &= (\nabla_{E_i}(\nabla_{E_i}A_V))X+(\nabla_{E_i}A_V)(\nabla_{E_i}X)\\
&\quad -c\langle A_V E_i,T\rangle(\langle E_i,T\rangle X-\langle X,T\rangle E_i)\\
&\quad +c\langle V,N\rangle(\langle A_N E_i,E_i\rangle X-\langle A_N X,E_i\rangle E_i)\\
&= C(E_i,E_i)X-c\langle A_V E_i,T\rangle(\langle E_i,T\rangle X-\langle X,T\rangle E_i)\\
&\quad +c\langle V,N\rangle(\langle A_N E_i,E_i\rangle X-\langle A_N X,E_i\rangle E_i),
\end{aligned} \tag{3.4}$$

where we used $\sigma(E_i,T)=-\nabla^{\perp}_{E_i}N$ and $\nabla_{E_i}T=A_N E_i$, which follow from the fact that ξ is parallel.

We also have

$$C(X,E_i)E_i=\nabla_X((\nabla_{E_i}A_V)E_i), \tag{3.5}$$

and, from (3.3), (3.4) and (3.5), we get

$$\begin{aligned}
C(E_i,E_i)X &= C(E_i,X)E_i+c\langle A_V E_i,T\rangle(\langle E_i,T\rangle X-\langle X,T\rangle E_i)\\
&\quad -c\langle V,N\rangle(\langle A_N E_i,E_i\rangle X-\langle A_N X,E_i\rangle E_i)\\
&= C(X,E_i)E_i+[R(E_i,X),A_V]E_i\\
&\quad +c\langle A_V E_i,T\rangle(\langle E_i,T\rangle X-\langle X,T\rangle E_i)\\
&\quad -c\langle V,N\rangle(\langle A_N E_i,E_i\rangle X-\langle A_N X,E_i\rangle E_i)
\end{aligned}$$

which means that

$$\begin{aligned} C(E_i,E_i)X &= \nabla_X((\nabla_{E_i}A_V)E_i)+[R(E_i,X),A_V]E_i \\ &\quad +c\langle A_VE_i,T\rangle(\langle E_i,T\rangle X-\langle X,T\rangle E_i) \\ &\quad -c\langle V,N\rangle(\langle A_NE_i,E_i\rangle X-\langle A_NX,E_i\rangle E_i). \end{aligned}$$

As A_V is symmetric, it follows that also $\nabla_{E_i}A_V$ is symmetric, and then, from (3.1), one obtains

$$\begin{aligned} \langle \textstyle\sum_{i=1}^2(\nabla_{E_i}A_V)E_i,Z\rangle &= \textstyle\sum_{i=1}^2\langle E_i,(\nabla_{E_i}A_V)Z\rangle=\sum_{i=1}^2\langle E_i,(\nabla_ZA_V)E_i\rangle \\ &\quad +c\langle V,N\rangle\textstyle\sum_{i=1}^2\langle E_i,\langle Z,T\rangle E_i-\langle E_i,T\rangle Z\rangle \\ &= \operatorname{trace}(\nabla_ZA_V)+c\langle V,N\rangle\langle T,Z\rangle \\ &= Z(\operatorname{trace}A_V)+c\langle V,N\rangle\langle T,Z\rangle \\ &= c\langle V,N\rangle\langle T,Z\rangle, \end{aligned}$$

for any vector Z tangent to Σ^2, since $\operatorname{trace}A_V=$ constant.

Therefore, at p, we have

$$\begin{aligned} (\nabla^2A_V)X &= \textstyle\sum_{i=1}^2C(E_i,E_i)X \\ &= c\nabla_X(\langle V,N\rangle T)+\textstyle\sum_{i=1}^2[R(E_i,X),A_V]E_i \\ &\quad +c\textstyle\sum_{i=1}^2\langle A_VE_i,T\rangle(\langle E_i,T\rangle X-\langle X,T\rangle E_i) \\ &\quad -c\textstyle\sum_{i=1}^2\langle V,N\rangle(\langle A_NE_i,E_i\rangle X-\langle A_NX,E_i\rangle E_i) \end{aligned}$$

and then, since $\bar\nabla_X\xi=0$ implies that $\sigma(X,T)=-\nabla^\perp_XN$ and $\nabla_XT=A_NX$,

$$\begin{aligned} (\nabla^2A_V)X &= \textstyle\sum_{i=1}^2[R(E_i,X),A_V]E_i+c\{2\langle V,N\rangle A_NX-\langle A_VX,T\rangle T \\ &\quad +\langle A_VT,T\rangle X-\langle X,T\rangle A_VT-2\langle H,N\rangle\langle V,N\rangle X\}. \end{aligned} \tag{3.6}$$

From the Gauss equation (2.2) of the surface Σ^2 and Lemma 3.1, we get

$$\begin{aligned} \textstyle\sum_{i=1}^2R(E_i,X)A_VE_i &= c\textstyle\sum_{i=1}^2\{\langle X,A_VE_i\rangle E_i-\langle E_i,A_VE_i\rangle X \\ &\quad -\langle X,T\rangle\langle A_VE_i,T\rangle E_i+\langle E_i,T\rangle\langle A_VE_i,T\rangle X \\ &\quad +\langle E_i,A_VE_i\rangle\langle X,T\rangle T-\langle X,A_VE_i\rangle\langle E_i,T\rangle T\} \\ &\quad +\textstyle\sum_{i=1}^2\sum_{\alpha=3}^{n+1}\{\langle A_\alpha X,A_VE_i\rangle A_\alpha E_i \\ &\quad -\langle A_\alpha E_i,A_VE_i\rangle A_\alpha X\}, \end{aligned}$$

which means that

$$\begin{aligned} \textstyle\sum_{i=1}^2R(E_i,X)A_VE_i &= c\{A_VX-(\operatorname{trace}A_V)X+(\operatorname{trace}A_V)\langle X,T\rangle T \\ &\quad -\langle A_VX,T\rangle T-\langle X,T\rangle A_VT+\langle A_VT,T\rangle X\} \\ &\quad +\textstyle\sum_{\alpha=3}^{n+1}\{A_VA_\alpha^2X-(\operatorname{trace}(A_VA_\alpha))A_\alpha X\}, \end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{2} A_V R(E_i,X)E_i &= c\sum_{i=1}^{2}\{\langle X,E_i\rangle A_V E_i - \langle E_i,E_i\rangle A_V X\\
&\quad -\langle X,T\rangle\langle E_i,T\rangle A_V E_i + \langle E_i,T\rangle\langle E_i,T\rangle A_V X\\
&\quad +\langle E_i,E_i\rangle\langle X,T\rangle A_V T - \langle X,E_i\rangle\langle E_i,T\rangle A_V T\}\\
&\quad +\sum_{i=1}^{2}\sum_{\alpha=3}^{n+1}\{\langle A_\alpha X,E_i\rangle A_V A_\alpha E_i\\
&\quad -\langle A_\alpha E_i,E_i\rangle A_V A_\alpha X\}\\
&= -c(1-|T|^2)A_V X\\
&\quad +\sum_{\alpha=3}^{n+1}\{A_V A_\alpha^2 X - (\operatorname{trace} A_\alpha)A_V A_\alpha X\}.
\end{aligned}$$

Finally, replacing in equation (3.6), we find

$$\begin{aligned}
(\nabla^2 A_V)X = \ & c\{(2-|T|^2)A_V X + 2\langle A_V T,T\rangle X - 2\langle A_V X,T\rangle T - 2\langle X,T\rangle A_V T\\
&+2\langle V,N\rangle A_N X - 2\langle H,N\rangle\langle V,N\rangle X\\
&-(\operatorname{trace} A_V)X + (\operatorname{trace} A_V)\langle X,T\rangle T\}\\
&+\sum_{\alpha=3}^{n+1}\{(\operatorname{trace} A_\alpha)A_V A_\alpha X - (\operatorname{trace}(A_V A_\alpha))A_\alpha X\},
\end{aligned}$$

and, after a straightforward computation,

$$\begin{aligned}
\langle\nabla^2 A_V, A_V\rangle &= \sum_{i=1}^{2}\langle(\nabla^2 A_V)E_i, A_V E_i\rangle\\
&= c\{(2-|T|^2)|A_V|^2 - 4|A_V T|^2 + 3(\operatorname{trace} A_V)\langle A_V T,T\rangle\\
&\quad +2(\operatorname{trace}(A_N A_V))\langle V,N\rangle - (\operatorname{trace} A_V)^2\\
&\quad -2(\operatorname{trace} A_V)\langle H,N\rangle\langle V,N\rangle\}\\
&\quad +\sum_{\alpha=3}^{n+1}\{(\operatorname{trace} A_\alpha)(\operatorname{trace}(A_V^2 A_\alpha)) - (\operatorname{trace}(A_V A_\alpha))^2\}.
\end{aligned}$$

Thus, from (3.2), we obtain the following

Proposition 3.2. *Let Σ^2 be an immersed surface in $M^n(c)\times\mathbb{R}$. If V is a normal vector field, parallel in the normal bundle, with* $\operatorname{trace} A_V = \text{constant}$*, then*

(3.7)

$$\begin{aligned}
\frac{1}{2}\Delta|A_V|^2 &= |\nabla A_V|^2 + c\{(2-|T|^2)|A_V|^2 - 4|A_V T|^2 + 3(\operatorname{trace} A_V)\langle A_V T,T\rangle\\
&\quad +2(\operatorname{trace}(A_N A_V))\langle V,N\rangle - (\operatorname{trace} A_V)^2\\
&\quad -2(\operatorname{trace} A_V)\langle H,N\rangle\langle V,N\rangle\}\\
&\quad +\sum_{\alpha=3}^{n+1}\{(\operatorname{trace} A_\alpha)(\operatorname{trace}(A_V^2 A_\alpha)) - (\operatorname{trace}(A_V A_\alpha))^2\},
\end{aligned}$$

where $\{E_\alpha\}_{\alpha=3}^{n+1}$ is a local orthonormal frame field in the normal bundle.

Corollary 3.3. *If Σ^2 is an immersed non-minimal pmc surface in $M^n(c)\times\mathbb{R}$ and ϕ_H is the operator defined by $\phi_H=\frac{1}{|H|}A_H-|H|\,\mathrm{I}$, then*

$$\begin{aligned}\tfrac{1}{2}\Delta|\phi_H|^2 &= |\nabla\phi_H|^2+\{c(2-3|T|^2)+4|H|^2-|\sigma|^2\}|\phi_H|^2\\ &\quad -2c|H|\langle\phi_H T,T\rangle+\tfrac{2c}{|H|}\langle H,N\rangle\,\mathrm{trace}(A_N\phi_H).\end{aligned}\tag{3.8}$$

Proof. From the definition of ϕ_H we have $\nabla\phi_H=\frac{1}{|H|}\nabla A_H$, $|\phi_H|^2=\frac{1}{|H|^2}|A_H|^2-2|H|^2$ and $\frac{1}{|H|^2}|A_HT|^2=\frac{1}{2}|T|^2|\phi_H|^2+|H|^2|T|^2+2|H|\langle\phi_HT,T\rangle$, where we used the fact that $|\phi_HT|^2=\frac{1}{2}|T|^2|\phi_H|^2$, which can be easily verified by working in a basis that diagonalizes ϕ_H and taking into account that $\mathrm{trace}\,\phi_H=0$.

Next, from equation (3.6), with $V=H$, we get $\langle\nabla^2A_H,\mathrm{I}\rangle=0$ and, therefore, from Proposition 3.2, it follows

$$\begin{aligned}\tfrac{1}{2}\Delta|\phi_H|^2 &= |\nabla\phi_H|^2+c(2-3|T|^2)|\phi_H|^2-2c|H|\langle\phi_HT,T\rangle\\ &\quad +\tfrac{2c}{|H|}\langle H,N\rangle\,\mathrm{trace}(A_N\phi_H)\\ &\quad +\textstyle\sum_{\alpha=3}^{n+1}\{(\mathrm{trace}\,A_\alpha)(\mathrm{trace}(\phi_H^2A_\alpha))-(\mathrm{trace}(\phi_HA_\alpha))^2\}.\end{aligned}\tag{3.9}$$

Now, let us consider the local orthonormal frame field $\{E_3=\frac{H}{|H|},E_4,\ldots,E_{n+1}\}$ in the normal bundle. One sees that $\mathrm{trace}\,A_3=2|H|$, $\mathrm{trace}\,A_\alpha=0$, $\alpha>3$, and hence

$$\sum_{\alpha=3}^{n+1}(\mathrm{trace}\,A_\alpha)(\mathrm{trace}(\phi_H^2A_\alpha))=2|H|(\mathrm{trace}(\phi_H^2A_3))=2(\mathrm{trace}(\phi_H^2A_H)).$$

From the definition of ϕ_H, we get $\phi_H^2A_H=|H|\phi_H^3+|H|^2\phi_H^2$ and, since $\mathrm{trace}\,\phi_H^3=0$,

$$2(\mathrm{trace}\,\phi_H^2A_H)=2|H|\,\mathrm{trace}\,\phi_H^3+2|H|^2\,\mathrm{trace}\,\phi_H^2=2|H|^2|\phi_H|^2.$$

We have just proved that

$$\sum_{\alpha=3}^{n+1}(\mathrm{trace}\,A_\alpha)(\mathrm{trace}(\phi_H^2A_\alpha))=2|H|^2|\phi_H|^2.\tag{3.10}$$

As we have seen in Lemma 3.1, since H is parallel, we have that A_H commutes with A_U, for any normal vector field U. Then, either there exists a basis that diagonalizes A_U, for all vectors U normal to Σ^2, or the surface is pseudo-umbilical, i.e. $A_H=|H|^2\,\mathrm{I}$. Moreover, since the map $p\in\Sigma^2\to(A_H-\mu\,\mathrm{I})(p)$, where μ is a constant, is analytic, it follows that if H is an umbilical direction, then this either holds on the whole surface or only on a closed set without interior points.

Since H is an umbilical direction everywhere implies that ϕ_H vanishes on the surface, and then (3.8) is verified, we will study only the case when H is an umbilical direction on a closed set without interior points, which means that H is not umbilical in an open dense set. We will work on this set and then will extend our result throughout Σ^2 by continuity.

Let $\{e_1,e_2\}$ be a basis that diagonalizes A_U, for all vectors U normal to the surface. Therefore, with respect to this basis, for $\alpha>3$, since $\mathrm{trace}\,A_\alpha=\mathrm{trace}\,\phi_H=0$, we have $A_\alpha=\begin{pmatrix}\mu_\alpha & 0\\ 0 & -\mu_\alpha\end{pmatrix}$ and $\phi_H=\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}$, and then

$$(\mathrm{trace}(\phi_HA_\alpha))^2=4a^2\mu_\alpha^2=|A_\alpha|^2|\phi_H|^2.\tag{3.11}$$

We also obtain $\phi_H A_3 = \phi_H^2 + |H|\phi_H$, which leads to

$$(\operatorname{trace}(\phi_H A_3))^2 = |\phi_H|^4 = (|A_3|^2 - 2|H|^2)|\phi_H|^2. \tag{3.12}$$

Finally, replacing (3.10), (3.11) and (3.12) in (3.9), we get equation (3.8). □

4. A Simons type equation and applications

In the following we shall derive a Simons type equation for non-minimal pmc surfaces in $M^3(c)\times\mathbb{R}$ and then we will use it in order to characterize some of these surfaces.

Everywhere in this section Σ^2 will be an immersed non-minimal pmc surface in a product space $M^3(c)\times\mathbb{R}$, with mean curvature vector H and the Gaussian curvature K.

Let us consider the local orthonormal frame field $\{E_3 = \frac{H}{|H|}, E_4\}$ in the normal bundle and denote by $\phi_3 = \phi_H = A_3 - |H|\,\mathrm{I}$ and $\phi_4 = A_4$. The normal part of ξ can be written as $N = \nu_3 E_3 + \nu_4 E_4$, where $\nu_3 = \langle \xi, E_3\rangle$ and $\nu_4 = \langle \xi, E_4\rangle$.

Since H is parallel in the normal bundle, it follows that also E_4 is parallel in the normal bundle. We use the same argument as in the proof of Corollary 3.3 to see that either H is an umbilical direction on the whole surface or it is not umbilical on an open dense set. In both cases, it is easy to verify that

$$(\operatorname{trace} A_3)(\operatorname{trace}(\phi_4^2 A_3)) = 2|H|^2|\phi_4|^2 \quad\text{and}\quad (\operatorname{trace}(\phi_4 A_3))^2 = (|A_3|^2 - 2|H|^2)|\phi_4|^2,$$

and then, since $(\operatorname{trace}\phi_4^2)^2 = |\phi_4|^4$, $\operatorname{trace}\phi_4 = 0$ and $|\phi_4 T|^2 = \frac{1}{2}|T|^2|\phi_4|^2$, from Proposition 3.2, we can derive the following formula for the Laplacian of $|\phi_4|^2$:

$$\frac{1}{2}\Delta|\phi_4|^2 = |\nabla\phi_4|^2 + \{c(2-3|T|^2) + 4|H|^2 - |\sigma|^2\}|\phi_4|^2 + 2c\nu_4 \operatorname{trace}(A_N\phi_4). \tag{4.1}$$

Next, let ϕ be the traceless part of the second fundamental form σ of the surface, given by

$$\phi(X,Y) = \sigma(X,Y) - \langle X,Y\rangle H = \sum_{\alpha=3}^{4}\langle \phi_\alpha X, Y\rangle E_\alpha.$$

We have $|\phi|^2 = |\phi_3|^2 + |\phi_4|^2 = |\sigma|^2 - 2|H|^2$ and then, using (3.8) and (4.1), we can state the following

Proposition 4.1. *If Σ^2 is an immersed non-minimal pmc surface in $M^3(c)\times\mathbb{R}$ and ϕ is the traceless part of its second fundamental form, then*

$$\begin{aligned} \tfrac{1}{2}\Delta|\phi|^2 = {} & |\nabla\phi_3|^2 + |\nabla\phi_4|^2 - |\phi|^4 + \{c(2-3|T|^2) + 2|H|^2\}|\phi|^2 \\ & -2c\langle\phi(T,T),H\rangle + 2c|\nu_3\phi_3 + \nu_4\phi_4|^2, \end{aligned} \tag{4.2}$$

or, equivalently,

$$\begin{aligned} \tfrac{1}{2}\Delta|\phi|^2 = {} & |\nabla\phi_3|^2 + |\nabla\phi_4|^2 - |\phi|^4 + \{c(2-3|T|^2) + 2|H|^2\}|\phi|^2 \\ & -2c\langle\phi(T,T),H\rangle + 2c|A_N|^2 - 4c\langle H,N\rangle^2. \end{aligned}$$

Theorem 4.2. *Let Σ^2 be an immersed complete non-minimal pmc surface in $\bar{M} = M^3(c)\times\mathbb{R}$, with $c<0$, such that*

$$\sup_{\Sigma^2}|\phi|^2 < 2|H|^2 + c(4-5|T|^2) \quad\text{and}\quad \langle\phi(T,T),H\rangle \geq 0.$$

Then $|\phi|^2 = 0$ and Σ^2 is a round sphere in $M^3(c)$.

Proof. First, using the Schwarz inequality, we get

$$|\nu_3\phi_3+\nu_4\phi_4|^2\leq(\nu_3^2+\nu_4^2)(|\phi_3|^2+|\phi_4|^2)=(1-|T|^2)|\phi|^2,$$

and this, together with (4.2) and the hypothesis, leads to

$$\begin{aligned}\tfrac{1}{2}\Delta|\phi|^2 &\geq -|\phi|^4+\{c(4-5|T|^2)+2|H|^2\}|\phi|^2-2c\langle\phi(T,T),H\rangle\\ &\geq\{-|\phi|^2+c(4-5|T|^2)+2|H|^2\}|\phi|^2 \qquad (4.3)\\ &\geq 0.\end{aligned}$$

On the other hand, we can prove that the Gaussian curvature K of the surface is bounded from below. Indeed, from (2.2), we have

$$2K=2c(1-|T|^2)+4|H|^2-|\sigma|^2=2c(1-|T|^2)+2|H|^2-|\phi|^2>c(3|T|^2-2)\geq c.$$

Therefore, since Σ^2 is complete, the Omori-Yau Maximum Principle holds on the surface. We take $u=|\phi|^2$ in Theorem 2.2 and it follows that there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}\subset\Sigma^2$ such that

$$\lim_{k\to\infty}|\phi|^2(p_k)=\sup_{\Sigma^2}|\phi|^2\quad\text{and}\quad\Delta|\phi|^2(p_k)<\frac{1}{k}.$$

From (4.3), we get that $\lim_{k\to\infty}|\phi|^2(p_k)=0$ and then $\sup_{\Sigma^2}|\phi|^2=0$, which implies $|\phi|^2=0$. Since $\phi_3=A_3-|H|\,\mathrm{I}=0$, it follows that H is an umbilical direction and this implies that Σ^2 is a totally umbilical surface in $M^3(c)$ (see Lemma 3 in [5]). Therefore, Σ^2 is a horosphere or a round sphere. Since $|\phi|^2<2|H|^2+4c$ implies that $K>-c>0$, Σ^2 cannot be flat, and then we conclude that the surface is a round sphere. □

Theorem 4.3. *Let Σ^2 be an immersed complete non-minimal pmc surface in $\bar{M}=M^3(c)\times\mathbb{R}$, with $c>0$, such that*

$$|\phi|^2\leq 2|H|^2+c(2-3|T|^2)\quad\text{and}\quad\langle\phi(T,T),H\rangle\leq 0.$$

Then ξ is normal to the surface and either

1. *$|\phi|^2=0$ and Σ^2 is a round sphere in $M^3(c)$, or*
2. *$|\phi|^2=2|H|^2+2c$ and Σ^2 is a torus $\mathbb{S}^1(r)\times\mathbb{S}^1(\sqrt{\frac{1}{c}-r^2})$, $r^2\neq\frac{1}{2c}$, in $M^3(c)$.*

Proof. From the Gauss equation (2.2) of the surface, since $|\phi|^2\leq 2|H|^2+c(2-3|T|^2)$, we obtain that

$$2K=2c(1-|T|^2)+4|H|^2-|\sigma|^2=2c(1-|T|^2)+2|H|^2-|\phi|^2\geq c|T|^2\geq 0$$

and then, a result of A. Huber in [13] implies that Σ^2 is a parabolic space.

On the other hand, from (4.2), we see that $\Delta|\phi|^2\geq 0$ and then $|\phi|^2$ is a bounded subharmonic function on a parabolic space. Therefore $|\phi|^2$ is a constant and, again using (4.2), we get that

$$\{-|\phi|^2+c(2-3|T|^2)+2|H|^2\}|\phi|^2=0,\quad\langle\phi(T,T),H\rangle=0\quad\text{and}\quad\nu_3\phi_3+\nu_4\phi_4=0.$$

Now we can split our study in two cases.

Case I: $|\phi|^2=0$. This case can be handled exactly like in the proof of Theorem 4.2.

Case II: $|\phi|^2=c(2-3|T|^2)+2|H|^2$. Since $|\phi|^2$ is a constant, it follows that $|T|^2$ is a constant and then that $\langle\nabla_X T,T\rangle=0$, for any vector X tangent to Σ^2. As $\bar{\nabla}_X\xi=0$ implies that $\nabla_X T=A_N X$, we have $\langle A_N X,T\rangle=0$. But $\nu_3\phi_3+\nu_4\phi_4=0$ means that $A_N=\langle H,N\rangle\,\mathrm{I}$, which implies that $\langle H,N\rangle\langle X,T\rangle=0$, for any tangent

vector X. Therefore, either ξ is orthogonal to Σ^2 at any point, or only on a closed set without interior points. In this second case, $\langle H,\xi\rangle = 0$ holds on an open dense set. In this set, we also have $\langle \bar{\nabla}_T H,\xi\rangle = -\langle A_H T, T\rangle = 0$. We have just shown that $\langle A_3 T, T\rangle = 0$ on an open dense set and, since we also know that

$$\langle \phi(T,T), H\rangle = |H|\langle \phi_3 T, T\rangle = |H|(\langle A_3 T, T\rangle - |H||T|^2) = 0,$$

one gets that $T = 0$ on the whole surface. Hence, Σ^2 lies in $M^3(c)$ and its Gaussian curvature is $K = \frac{c}{2}|T|^2 = 0$. We use a similar argument to that in the proof of Theorem 1.5 in [3] (see also [15]) to conclude. □

5. A gap Theorem for pmc surfaces with non-negative Gaussian curvature

In this Section we will prove our main results, Theorem 5.2 and Theorem 5.3. In order to do that, let us consider an immersed pmc surface Σ^2 in $M^n(c)\times\mathbb{R}$, and first we shall compute the Laplacian of $|T|^2$.

Let $\{e_1, e_2\}$ be an orthonormal in $T_p\Sigma^2$, $p\in\Sigma^2$, extend e_1 and e_2 to vector fields E_1 and E_2 in a neighborhood of p such that $\nabla E_i = 0$ at p. At p, we have

$$\begin{aligned}
\tfrac{1}{2}\Delta|T|^2 &= \textstyle\sum_{i=1}^2(\langle \nabla_{E_i}T, \nabla_{E_i}T\rangle + \langle \nabla_{E_i}\nabla_{E_i}T, T\rangle)\\
&= |A_N|^2 + \textstyle\sum_{i=1}^2\langle \nabla_{E_i}A_N E_i, T\rangle
\end{aligned}$$

and, since $\nabla_X A_N$ is symmetric,

$$\begin{aligned}
\textstyle\sum_{i=1}^2\langle \nabla_{E_i}A_N E_i, T\rangle &= \textstyle\sum_{i=1}^2\langle (\nabla_{E_i}A_N)E_i, T\rangle\\
&= \textstyle\sum_{i=1}^2\langle (\nabla_{E_i}A_N)T, E_i\rangle = \textstyle\sum_{i=1}^2\langle \nabla_{E_i}A_N T - A_N\nabla_{E_i}T, E_i\rangle\\
&= \textstyle\sum_{i=1}^2\langle \nabla_{E_i}\nabla_T T - \nabla_{\nabla_{E_i}T}T, E_i\rangle\\
&= \textstyle\sum_{i=1}^2\langle \nabla_{E_i}\nabla_T T + \nabla_{[T,E_i]}T, E_i\rangle\\
&= \textstyle\sum_{i=1}^2(\langle \nabla_T\nabla_{E_i}T, E_i\rangle - \langle R(T,E_i)T, E_i\rangle)\\
&= |T|^2K + \textstyle\sum_{i=1}^2\langle \nabla_T A_N E_i, E_i\rangle\\
&= |T|^2K + \textstyle\sum_{i=1}^2 T(\langle A_N E_i, E_i\rangle) = |T|^2K + T(\operatorname{trace} A_N)\\
&= |T|^2K + 2T(\langle H, N\rangle) = |T|^2K - 2\langle \sigma(T,T), H\rangle\\
&= c|T|^2(1-|T|^2) - \tfrac{1}{2}|T|^2|\phi|^2 - 2\langle \phi(T,T), H\rangle - |T|^2|H|^2,
\end{aligned}$$

where we used $\nabla_X T = A_N X$ and $\nabla^{\perp}_X N = -\sigma(X,T)$, and ϕ is the traceless part of the second fundamental form σ of the surface.

Now, we can state

Proposition 5.1. *If Σ^2 is an immersed pmc surface in $M^n(c)\times\mathbb{R}$, then*

$$\frac{1}{2}\Delta|T|^2 = |A_N|^2 - \frac{1}{2}|T|^2|\phi|^2 - 2\langle \phi(T,T), H\rangle + c|T|^2(1-|T|^2) - |T|^2|H|^2. \tag{5.1}$$

In [4] the authors introduced a holomorphic differential, defined on pmc surfaces in $M^n(c)\times\mathbb{R}$ (when $n = 2$ this is just the Abresch-Rosenberg differential, defined in

[1]). This holomorphic differential is the $(2,0)$-part of the quadratic form Q, given by

$$Q(X,Y) = 2\langle\sigma(X,Y),H\rangle - c\langle X,\xi\rangle\langle Y,\xi\rangle.$$

Using this result and Proposition 5.1, we can characterize pmc 2-spheres Σ^2 immersed in $M^n(c)\times\mathbb{R}$, whose second fundamental form satisfies a certain condition.

Theorem 5.2. *Let Σ^2 be an immersed pmc 2-sphere in $M^n(c)\times\mathbb{R}$, such that*

(1) $|T|^2 = 0$ *or* $|T|^2 \geq \frac{2}{3}$ *and* $|\sigma|^2 \leq c(2-3|T|^2)$, *if* $c<0$;
(2) $|T|^2 \leq \frac{2}{3}$ *and* $|\sigma|^2 \leq c(2-3|T|^2)$, *if* $c>0$.

Then, Σ^2 is either a minimal surface in a totally umbilical hypersurface of $M^n(c)$ or a standard sphere in $M^3(c)$.

Proof. If ξ is orthogonal to the surface in an open connected subset, then this subset lies in $M^n(c)$, and by analyticity, it follows that Σ^2 lies in $M^n(c)$. In this case we use Theorem 4 in [18] to conclude.

Next, let us assume that we are not in the previous case. Then, we can choose an orthonormal frame $\{e_1,e_2\}$ on the surface, such that $e_1 = \frac{T}{|T|}$. Since Σ^2 is a sphere and the $(2,0)$-part of Q is holomorphic, it follows that it vanishes on the surface. This means that $Q(e_1,e_1) = Q(e_2,e_2)$ and $Q(e_1,e_2) = 0$. From $Q(e_1,e_1) = Q(e_2,e_2)$, we get $2\langle\sigma(e_1,e_1)-\sigma(e_2,e_2),H\rangle = c|T|^2$. We have

$$\langle\phi(T,T),H\rangle = \langle\sigma(T,T),H\rangle - |T|^2|H|^2 = \frac{1}{2}|T|^2\langle\sigma(e_1,e_1)-\sigma(e_2,e_2),H\rangle = \frac{1}{4}c|T|^4,$$

and then (5.1) becomes

$$\frac{1}{2}\Delta|T|^2 = |A_N|^2 + \frac{1}{2}|T|^2(-|\sigma|^2 + c(2-3|T|^2)) \geq 0.$$

Since Σ^2 is a sphere and $|T|^2$ is a bounded subharmonic function, it results that $|T|^2$ is constant, and then that $|A_N|^2 = 0$ and $|T|^2(-|\sigma|^2 + c(2-3|T|^2)) = 0$. Since $A_N = 0$ and ξ is parallel, we obtain that $\nabla_X T = 0$, for any tangent vector X, which implies that $K=0$. Since our surface is a sphere, this is a contradiction. Therefore, Σ^2 lies in $M^n(c)$ and, again using Theorem 4 in [18] (see also Theorem 2 in [5]) we come to the conclusion. $\square$

Next, assume that Σ^2 is an immersed non-minimal pmc surface in $M^3(c)\times\mathbb{R}$. Then, from (4.2) and (5.1), we obtain

(5.2)
$$\begin{aligned}\tfrac{1}{2}\Delta(|\phi|^2 - c|T|^2) \;=\;& |\nabla\phi_3|^2 + |\nabla\phi_4|^2 + \{-|\phi|^2 + \tfrac{c}{2}(4-5|T|^2) + 2|H|^2\}|\phi|^2\\ & + c|A_N|^2 - 4c\langle H,N\rangle^2 + c|T|^2|H|^2 - c^2|T|^2(1-|T|^2).\end{aligned}$$

Using this equation, we shall prove the following

Theorem 5.3. *Let Σ^2 be an immersed complete non-minimal pmc surface in $\bar{M} = M^3(c)\times\mathbb{R}$, with $c>0$. Assume*

i) $|\phi|^2 \leq 2|H|^2 + 2c - \frac{5c}{2}|T|^2$, *and*
ii) a) $|T| = 0$, *or*
  b) $|T|^2 > \frac{2}{3}$ *and* $|H|^2 \geq \frac{c|T|^2(1-|T|^2)}{3|T|^2-2}$.

Then either

(1) $|\phi|^2 = 0$ *and Σ^2 is a round sphere in $M^3(c)$, or*
(2) $|\phi|^2 = 2|H|^2 + 2c$ *and Σ^2 is a torus $\mathbb{S}^1(r)\times\mathbb{S}^1(\sqrt{\frac{1}{c}-r^2})$, $r^2 \neq \frac{1}{2c}$, in $M^3(c)$.*

Proof. If $|T|^2 = 0$, it is easy to see that (5.2) implies

$$\frac{1}{2}\Delta(|\phi|^2 - c|T|^2) \geq \{-|\phi|^2 + 2c + 2|H|^2\}|\phi|^2 \geq 0.$$

Next, let us assume that $|T|^2 > \frac{2}{3}$ and $|H|^2 \geq \frac{c|T|^2(1-|T|^2)}{3|T|^2-2}$. Since

$$|A_N|^2 - 2\langle H, N\rangle = |\nu_3\phi_3 + \nu_4\phi_4|^2 \geq 0$$

and, from the Schwarz inequality, we have

$$\langle H, N\rangle^2 \leq |N|^2|H|^2 = (1 - |T|^2)|H|^2, \tag{5.3}$$

then, from (5.2), it follows that

$$\begin{aligned} \tfrac{1}{2}\Delta(|\phi|^2 - c|T|^2) &\geq \{-|\phi|^2 + \tfrac{c}{2}(4 - 5|T|^2) + 2|H|^2\}|\phi|^2 \\ &\quad + c(3|T|^2 - 2)|H|^2 - c^2|T|^2(1 - |T|^2) \\ &\geq 0. \end{aligned}$$

The Gaussian curvature of the surface satisfies

$$2K = 2c(1 - |T|^2) + 2|H|^2 - |\phi|^2 \geq \frac{1}{2}c|T|^2 \geq 0,$$

which means that Σ^2 is a parabolic space. Now, since $|\phi|^2 - c|T|^2$ is a bounded subharmonic function, it follows that it is constant. Therefore, either $|\phi|^2 = 0$ or $|\phi|^2 = 2|H|^2 + 2c - \frac{5c}{2}|T|^2$. The first case can be handled exactly as in the proof of Theorem 4.2. As for the second case, from (5.2), we have that $\nu_3\phi_3 + \nu_4\phi_4 = 0$, which means that $A_N = \langle H, N\rangle \mathrm{I}$, and the equality holds in (5.3), i.e. either $N = \nu_3 H$ or ξ is tangent to the surface. If ξ is tangent to the surface, from (5.2) and the hypothesis, it follows that Σ^2 is a minimal surface, which is a contradiction. Hence, $N = \nu_3 H$ and we get $A_H = |H|^2 \mathrm{I}$, which, again using Lemma 3.1 in [5], implies that the surface lies in $M^3(c)$. We then come to the conclusion in the same way as in the proof of the second part of Theorem 4.3. □

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